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On Non-uniform Convergence of Infinite Series.

By A. S. Chessin.

A letter received by the author since the appearance of his note "On a Point of the Theory of Functions,"[*] shows that his statement is liable to be misunderstood, and it is therefore desirable to bring the point out clearer.

The aim was simply to call attention to the fact that between non-uniform convergence and discontinuity there is a connection which has not yet been brought to light, but was rather hidden by the help of an artifice. The proposition enunciated in the note referred to is not intended to be a new theorem; indeed it is nothing but the enunciation of the simple fact that the product of a continuous function and a discontinuous one may be a continuous function. For example, if $\phi(x) = a$ for all values of x in a given interval except for $x = x_0$ and $\phi(x_0) = b \neq a$, the numbers a and b being finite, $(x - x_0)\phi(x)$ will be continuous at $x = x_0$. The character of the function $\phi(x)$ is hidden in the product $(x - x_0)\phi(x)$. So it is with non-uniformly convergent series of the type considered in my previous note. Let

$$f(x) = \sum_1^\infty f_m(x)$$

be convergent for all the values of x in the given interval except for $x = x_0$, for which value this series need not be convergent. The non-uniformly convergent series

$$F(x) = \sum_1^\infty \theta(x) f_m(x)$$

considered in my previous note represents a continuous function if an arbitrarily small but finite number h can be found such that

$$|\theta(x_0 \pm \eta h) f(x_0 \pm \eta h)| < \varepsilon, \tag{1}$$

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[*] See Vol. XVIII, No. 1, p. 98.

where $0 < \eta \overset{=}{<} 1$. This condition is necessary and sufficient. One way of satisfying this condition would be in having $\sum_{1}^{\infty} f_m(x)$ finite in the neighborhood of x_0, whatever be its value for $x = x_0$; another would still remain in letting $f(x)$ tend to become infinite (in a determinate or indeterminate way) as x approaches the value x_0. But this case is reducible to the one just considered. In fact if we put $\phi_m(x) = \frac{f_m(x)}{f(x)}$ and $\vartheta(x) = f(x)\,\theta(x)$, the series

$$\sum_{1}^{\infty} \vartheta(x)\,\phi_m(x)$$

will come into the class for which $\sum_{1}^{\infty} \phi_m(x)$ remains finite in the neighborhood of $x = x_0$.

If, on the other hand, such a number h for which the relation (1) would be satisfied could not be found, the function $F(x)$ would be discontinuous. But this is only possible if $\sum_{1}^{\infty} f_m(x)$ tends to become infinite (in a determinate or indeterminate way), while a reduction to the first class is no more possible.

Dr. W. F. Osgood, of Harvard University, has truly called my attention to the fact that, adding to the series $\sum_{1}^{\infty} \theta(x) f_m(x)$ a convergent numerical series, $\sum_{1}^{\infty} \alpha_m$, we obtain again a non-uniformly convergent series $\sum_{1}^{\infty} \{\alpha_m + \theta(x) f_m(x)\}$ which represents a continuous function if $F(x)$ is continuous; and yet this series is not of the form $\sum_{1}^{\infty} \theta(x) f_m(x)$. More generally this can be said about the series

$$\sum_{1}^{\infty}{}^{m} \{\chi_m(x) + \theta(x) f_m(x)\}, \tag{2}$$

where $\sum_{1}^{\infty}{}^{m} \chi_m(x)$ is uniformly convergent. However, the connection between the discontinuity of functions and non-uniform convergence of series is as much concealed in the last series as in those considered before. The fact is, that in all examples that I am aware of, the construction is based on non-uniformly convergent series representing *discontinuous* functions. This seems to indicate that the connection between non-uniform convergence of infinite series and discontinuity of functions is not a casual one, and it would be of the greatest interest to have more light thrown on this point.